Non-Linear Quantization of Integrable Classical Systems

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Abstract

It is demonstrated that the so-called "unavoidable quantum anomalies" can be avoided in the framework of a special non-linear quantization scheme. In this scheme, the quantized hamiltonians are represented by non-linear but homogeneous operators in Hilbert space. The nonlinear terms are of the same order as quantum anomalies, and their role is to cancel anomalies. The quantization method proposed is applicable to integrable classical dynamical systems and the result of quantization is again an integrable (but, generally, non-linear) "quantum" system. A simple example is discussed in detail. Irrespective of the existence of possible physical applications, the method provides a constructive way for extending the notion of quantum integrability to non-linear spectral problems and gives a practical tool for buiding completely integrable non-linear spectral equations in Hilbert space.

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1 Introduction

It is known that quantization does not generally preserve zero commutation relations between observables. In other words, classically integrable systems may cease to be integrable after their quantization. Of course, the quantization procedure is not unique. There are infinitely many ways to quantize a system and, for this reason, the integrability can sometimes be recovered after an appropriate choice of a quantization scheme. However, the typical situation (especially for multi-dimensional systems) is that there are no quantization schemes at all in which such a recovering would be possible. In such a case we speak of the "unavoidable quantum anomalies".

In order to understand the reason for the existence of unavoidable quantum anomalies, consider a certain completely integrable N-dimensional classical system with N mutually commuting integrals of motion $H_n,\ n = 1, \dots, N$. Denote by $\hat{H}_n^{(0)},\ n = 1, \dots, N$ the quantum versions of these classical integrals of motion obtained within some fixed quantization scheme. Assume that operators $\hat{H}_n^{(0)}$ do not commute with each other. For a randomly chosen quantization scheme this is a typical situation. Computing the commutator of these operators, we obtain

$$[\hat{H}_n^{(0)}, \hat{H}_m^{(0)}] = \hbar^2 \hat{F}_{nm}^{(1)} + \hbar^3 \hat{F}_{nm}^{(2)} + \dots . \tag{1.1}$$

Obviously, this expansion should start with $\hbar^2$. Otherwise we would obtain non-zero Poisson brackets in the classical limit, which is impossible because $H_n,\ n = 1, \dots, N$ are, by assumption, integrals of motion. Now remember that the chosen quantization scheme is not unique. This is a trivial consequence of the fact that, instead of operators $\hat{H}_n^{(0)},\ n = 1, \dots, N$, we could consider arbitrary operators of the form

$$\hat{H}_n = \hat{H}_n^{(0)} + \hbar \hat{H}_n^{(1)} + \hbar^2 \hat{H}_n^{(2)} + \dots, \qquad n = 1, \dots, N. \tag{1.2}$$

which, evidently, would have the same classical limit. This fact suggests to improve the commutation relations (1.1) by replacing the operators $\hat{H}_n^{(0)}$ by more general operators (1.2) with appropriately chosen corrections $\hat{H}_n^{(1)}, \hat{H}_n^{(2)}, \dots$. Requiring that $[\hat{H}_n, \hat{H}_m] = 0$, using (1.1) and (1.2) and collecting the terms of the same order in $\hbar$, we obtain the system of recurrence relations for corrections of our interest. For example, the equation determining the first correction reads

$$[\hat{H}_n^{(0)}, \hat{H}_m^{(1)}] + [\hat{H}_n^{(1)}, \hat{H}_m^{(0)}] + \hbar \hat{F}_{nm}^{(1)} = 0, \qquad n > m, \qquad n, m = 1, \dots, N. \tag{1.3}$$

However, even this first equation clearly demonstrates that the solution of the problem does not generally exist. Indeed, the number of unknowns in equation (1.3) is N. At the same time, the number of equations is $N(N-1)/2$. If $N > 3$, i.e., for more than three dimensional models, the system becomes overdetermined and generally has no solutions. This is what we meant saying that for multi-dimensional models the situation with quantum anomalies is typical and cannot be avoided in the framework of standard formalizm of quantum mechanics[5].

The aim of the present paper is to demonstrate that construction of quantization schemes free of any quantum anomalies (at least on the states) becomes possible if one drops out the condition of linearity of quantum mechanics. The resulting non-linear Schrödinger equations are

[5]Of course, in the presence of some symmetry in a model, the equations for corrections may cease to be independent and their effective number may decrease. In this case the quantization without anomalies may become possible. Such situation is usually realized in models obtainable in the framework of the r-matrix method if the underlying symmetry is $su(2)$ or $su(3)$. As much as we know, for higher symmetries the problem with quantum anomalies in r-matrix approach remains still unsolved.

very simalar to Doebner – Goldin equations [1] recently derived in the framework of a rather general quantization scheme based on the use of infinite-dimensional algebras of vector fields and group of diffeomorphisms [2, 3]. Speaking of non-linear Schrödinger equations and non-linear quantum mechanics, we mean a hypothetic quantum theory in which the observables are represented by non-linear operators in Hilbert space. At the present time there are many examples of such theories (see e.g. refs. [4, 5, 6, 7, 1] and references therein). Because up to now there are no experemental indications that the quantum world is non-linear, the measure of non-linear effects in all reasonable theories should be very small. In our scheme this condition is automatically satisfied because in this scheme the role of non-linear terms is to cancel quantum anomalies, and therefore, the order of these terms is the same as the order of anomalies, i.e., at most $\hbar^2$.

In this paper we restrict ourselves to discussing a particular (but very important) case of quantization of completely integrable classical systems. We leave the consideration of the general case to later publications. In the case of completely integrable systems, the procedure of quantization consists of three steps: 1) separation of variables in an integrable classical system[6], 2) standard quantization of one-dimensional equations so obtained, and 3) reconstruction of an integrable quantum system from the quantized one-dimensional equations.

The first step is most non-trivial from the practical point of view, because, in practice, it is often very difficult to check whether a given classical system is integrable or not, and even if it is, in which canonical coordinates it is separable. However, this difficulty can be avoided by using the "classical inverse method of separation of variables" (classical IMSV), when instead of checking the integrability or separability of a given system, we are simply building systems which are separable and integrable by construction [8, 9].

The last step is nothing else than the so-called "quantum inverse method of separation of variables" (quantum IMSV), which, up to now, has been applied to systems of separated equations with linear dependence on separation constants (see e.g. refs. [9, 10, 11, 12]). For such equations the resulting quantum systems are always linear. The non-linear quantum operators appear only if the separated equations depend on separation constants non-linearly.

These two classical and quantum versions of IMSV will be discussed in detail in the two subsequent sections 2 and 3. In section 4 we explain what we mean of non-linear quantization. The last section 5 is devoted to discussion of specific two-dimensional models for which all the aspects of the proposed quantization scheme become transparent and clear. Unfortunately, these models, because of their two-dimensionality, are not the examples of models with true quantum anomalies. The discussion of the latter would force us to work in at least four-dimensional space, which, of course would lead to very cumbersome expressions, non-desirable in the framework of this paper.

2 Classical version of IMSV

Let $F_i = F_i(x, y, z_1, \ldots, z_N),\ i = 1, \ldots, N$ be N (may be partially or completely coinciding) polynomial functions of $N + 2$ complex variables $x, y, z_1, \ldots, z_N$. Consider N ordinary first-

[6]The separation of variables is understood here in the generalized (Sklyanin) sense. Following Sklyanin [8], we call a N-dimensional classical system separable if there exist such a *canonical transformation* to canonically conjugated variables $p_i, q_i,\ i = 1, \ldots, N$ in which the Hamilton – Jacobi equation for the system becomes equivalent to a system of one-dimensional multi-parameter spectral equations $F_i(q_i, p_i, h_1, \ldots, h_N),\ i = 1, \ldots, N$ with some polynomial functions F_i. For more details see ref. [8].

order differential equations

$$F_i\left(q_i, \frac{\partial S_i(q_i)}{\partial q_i}, h_1, \dots, h_N\right) = 0, \quad i = 1, \dots, N \tag{2.1}$$

for N functions $S_i(q_i),\ i = 1, \dots, N$. Obviously, all these equations can be explicitly integrated after solving the algebraic equations $F_i(x, y, z_1, \dots, z_N) = 0$ with respect to the second variable, $y = Y_i(x, z_1, \dots, z_N)$. We shall write the result in the following form

$$S_i(q_i) = \int Y_i(q_i, h_1, \dots, h_N)\mathrm{d}q_i, \quad i = 1, \dots, N \tag{2.2}$$

stressing the fact that all solutions depend on N arbitrary numbers $h_1, \dots, h_N$ parametrizing equations (2.1).

Let us now note that the equalities (2.1) are not violated if we replace the functions $S_i(q_i)$ in them by their sum

$$S(\mathbf{q}) = \sum_{k=1}^{N} S_k(q_k). \tag{2.3}$$

This gives us the following new system of equations

$$F_i\left(q_i, \frac{\partial S(\mathbf{q})}{\partial q_i}, h_1, \dots, h_N\right) = 0, \quad i = 1, \dots, N, \tag{2.4}$$

for a single function $S(\mathbf{q})$ of N variables $q_1, \dots, q_N$. This system admits (by construction) a total separation of variables and its solution is given by formulas (2.3) and (2.2).

At the same time, the system (2.4) can be interpreted as a system of N algebraic equations for N unknown variables $h_1, \dots, h_N$. Solving it with respect to $h_1, \dots, h_N$ we obtain

$$h_\alpha = H_\alpha\left(\mathbf{q}, \frac{\partial S(\mathbf{q})}{\partial \mathbf{q}}\right), \quad \alpha = 1, \dots, N, \tag{2.5}$$

where $H_\alpha(\mathbf{q}, \mathbf{p})$ are some functions of two N-component vector variables $\mathbf{q} = \{q_1, \dots, q_N\}$ and $\mathbf{p} = \{p_1, \dots, p_N\}$. Interpreting these variables as canonically conjugated coordinates and momenta, $\{q_\alpha, p_\beta\} = \delta_{\alpha\beta}$, let us consider a N-dimensional classical dynamical system with the hamiltonian

$$H(\mathbf{q}, \mathbf{p}) = E(H_1(\mathbf{q}, \mathbf{p}), \dots, H_r(\mathbf{q}, \mathbf{p})) \tag{2.6}$$

in which $E(h_1, \dots, h_N)$ is an arbitrarily fixed function of $h_1, \dots, h_N$.

It is not difficult to see that the stationary Hamilton–Jacobi equation for the hamiltonian $H(\mathbf{q}, \mathbf{p})$,

$$H\left(\mathbf{q}, \frac{\partial S(\mathbf{q})}{\partial \mathbf{q}}\right) = E, \tag{2.7}$$

is separable by construction and its complete solution (i.e. a solution parametrized by N arbitrary parameters) has the form

$$E = E(h_1, \dots, h_N), \quad S(\mathbf{q}) = \sum_{i=1}^{N} \int Y_i(q_i, h_1, \dots, h_N)\mathrm{d}q_i. \tag{2.8}$$

This immediately follows from the fact that, after fixing the form of the function $E(h_1, \ldots, h_N)$, the equation (2.7) becomes equivalent to the separable system (2.6).

So we have demonstrated that any set of N functions $F_i(x, y, z_1, \ldots, z_N)$, $i = 1, \ldots N$ generates a certain separable classical dynamical system by means of the procedure which we shall refer below to as the *classical inverse procedure of separation of variables*.

It is not difficult to show that the classical system obtained in such a way is not only separable but also completely integrable in the sense that its hamiltonian admits enough mutually commuting integrals of motion. In order to demonstrate this fact, it is sufficient to show that functions $H_\alpha(\mathbf{q}, \mathbf{p})$, $\alpha = 1, \ldots, N$ form a commutative family.

Indeed, if $H_\alpha = H_\alpha(\mathbf{q}, \mathbf{p}), \alpha = 1, \ldots N$, then we have by definition

$$F_i = F_i(q_i, p_i, H_1, \ldots, H_N) = 0, \quad i = 1, \ldots, N. \tag{2.9}$$

Therefore

$$\frac{dF_i}{dq_k} = \frac{\partial F_i}{\partial q_k} + \sum_{l=1}^{N} \frac{\partial F_\alpha}{\partial H_l} \frac{\partial H_l}{\partial q_k} = 0, \tag{2.10}$$

$$\frac{dF_i}{dp_k} = \frac{\partial F_i}{\partial p_k} + \sum_{l=1}^{N} \frac{\partial F_i}{\partial H_l} \frac{\partial H_l}{\partial p_k} = 0, \tag{2.11}$$

and, consequently,

$$\sum_{l=1}^{N} \frac{\partial F_i}{\partial H_l} \frac{\partial H_l}{\partial q_k} = -\frac{\partial F_i}{\partial q_k}, \tag{2.12}$$

$$\sum_{l=1}^{N} \frac{\partial F_i}{\partial H_l} \frac{\partial H_l}{\partial p_k} = -\frac{\partial F_i}{\partial p_k}. \tag{2.13}$$

On the other hand, we have

$$\begin{aligned} 0 = \{F_{i_1}, F_{i_2}\} = \sum_{k=1}^{N} \left[\frac{dF_{i_1}}{dq_k} \frac{dF_{i_2}}{dp_k} - \frac{dF_{i_1}}{dp_k} \frac{dF_{i_2}}{dq_k} \right] = \\ = \sum_{k=1}^{N} \left[\frac{\partial F_{i_1}}{\partial q_k} \frac{\partial F_{i_2}}{\partial p_k} - \frac{\partial F_{i_1}}{\partial p_k} \frac{\partial F_{i_2}}{\partial q_k} \right] + \\ + \sum_{k,l=1}^{N} \left[\frac{\partial F_{i_1}}{\partial q_k} \frac{\partial F_{i_2}}{\partial H_l} \frac{\partial H_l}{\partial p_k} - \frac{\partial F_{i_1}}{\partial p_k} \frac{\partial F_{i_2}}{\partial H_l} \frac{\partial H_l}{\partial q_k} \right] + \\ + \sum_{k,l=1}^{N} \left[\frac{\partial F_{i_1}}{\partial H_l} \frac{\partial H_l}{\partial q_k} \frac{\partial F_{i_2}}{\partial p_k} - \frac{\partial F_{i_1}}{\partial H_l} \frac{\partial H_l}{\partial p_k} \frac{\partial F_{i_2}}{\partial q_k} \right] + \\ + \sum_{k,l=1}^{N} \frac{\partial F_{i_1}}{\partial H_k} \frac{\partial F_{i_2}}{\partial H_l} \{H_k, H_l\}. \end{aligned} \tag{2.14}$$

Applying (2.12) and (2.13) to (2.14) we obtain

$$\sum_{k,l=1}^{N} \frac{\partial F_{i_1}}{\partial H_k}\frac{\partial F_{i_2}}{\partial H_l}\{H_k, H_l\} = \sum_{k=1}^{N}\left[\frac{\partial F_{i_1}}{\partial q_k}\frac{\partial F_{i_2}}{\partial p_k} - \frac{\partial F_{i_1}}{\partial p_k}\frac{\partial F_{i_2}}{\partial q_k}\right]. \tag{2.15}$$

Since the functions F_{i_1} and F_{i_2} depend on different variables if $i_1 \neq i_2$, we have

$$\sum_{k,l=1}^{N} \frac{\partial F_{i_1}}{\partial H_k}\frac{\partial F_{i_2}}{\partial H_l}\{H_k, H_l\} = 0, \tag{2.16}$$

for all i_1 and i_2, and, because of the invertibility of the matrix $||\partial F_i/\partial H_k||$, we have

$$\{H_k, H_l\} = 0. \tag{2.17}$$

It is not difficult to see that the functions $H_1, \ldots, H_N$ constructed in such a way are functinally independent and thus the classical system with hamiltonian $H = E(H_1, \ldots, H_N)$ is completely integrable.

3 Quantum version of IMSV

Let $F_i = F_i(x, y, z_1, \ldots, z_N),\ i = 1, \ldots, N$ be now N (may be partially or completely coinciding) polynomial functions of $N+2$ generally non-commuting variables $x, y, z_1, \ldots, z_N$. We assume that these variables are distributed in the expression in that order in which they are written. Consider N linear differential equations

$$F_i\left(q_i, \mathrm{i}\hbar\frac{\partial}{\partial q_i}, h_1, \ldots, h_N\right)\Psi_i(q_i) = 0, \quad i = 1, \ldots, N \tag{3.1}$$

for N functions $\Psi_i(q_i), i = 1, \ldots, N$. We shall call equations of the type (3.1) *multi-parameter spectral equations*. The role of the spectral parameters in them is played by the numbers $h_1, \ldots, h_N$. The problem is to find all admissible values of these parameters for which the system (3.1) has solutions belonging to a certain *a priori* given classes W_i of functions $\Psi_i(q_i) \in W_i,\ i = 1, \ldots, N$. The set of all admissible "N-plets" $\{h_1, \ldots, h_N\}$ will be called the *spectrum* of the system (3.1). Below we shall assume that the classes $W_i,\ i = 1, \ldots, N$ are chosen in such a way that the system (3.1) has a discrete spectrum. The corresponding discrete set of solutions we represent as

$$\Psi_i(q_i) = \Xi_i(q_i, h_1, \ldots, h_N), \quad i = 1, \ldots, N, \tag{3.2}$$

stressing their correspondence to the values of spectral parameters $h_1, \ldots, h_N$.

Let us now note that the equalities (3.2) are not violated if we replace the functions $\Psi_i(q_i)$ in them by their product

$$\Psi(\mathbf{q}) = \prod_{i=1}^{N}\Psi_\alpha(q_i) \tag{3.3}$$

belonging to the class $W = \otimes_{i=1}^{N} W_i$. This gives us the following new system of equations

$$F_i\left(q_i, \mathrm{i}\hbar\frac{\partial}{\partial q_i}, h_1, \ldots, h_N\right)\Psi(\mathbf{q}) = 0, \quad i = 1, \ldots, N, \tag{3.4}$$

for a single function $\Psi(\mathbf{q})$ of the N-component vector variable $\mathbf{q} = \{q_1, \dots, q_N\}$. This system admits (by construction) a total separation of variables and its solution is given by formulas (3.3) and (3.2).

At the same time, the system (3.4) can be interpreted as a system of N algebraic equations for N unknown spectral parameters $h_1, \dots, h_N$. Solving it with respect to $h_1, \dots, h_N$ and multiplying the result by $\Psi(\mathbf{q})$ we obtain

$$h_\alpha \Psi(\mathbf{q}) = \hat{H}_\alpha \left(\mathbf{q}, \mathrm{i}\hbar \frac{\partial}{\partial \mathbf{q}} \right) \Psi(\mathbf{q}), \quad \alpha = 1, \dots, N, \tag{3.5}$$

where $\hat{H}_\alpha(\mathbf{q}, \mathbf{p})$ is some formal writing for, generally, non-linear differential operators built from two N-component non-commuting operators $\mathbf{q} = \{q_1, \dots, q_N\}$ and $\mathbf{p} = \{\mathrm{i}\hbar\partial/\partial q_1, \dots, \mathrm{i}\hbar\partial/\partial q_N\}$. We interpret these operators as operators of coordinates and momenta satisfying the Heisenberg commutation relations $[\mathbf{q} \otimes \mathbf{p}] = \mathrm{i}\hbar\mathbf{I}$.

Consider the N-dimensional non-linear operator

$$\hat{H}(\mathbf{q}, \mathbf{p}) = E(\hat{H}_1(\mathbf{q}, \mathbf{p}), \dots, \hat{H}_r(\mathbf{q}, \mathbf{p})) \tag{3.6}$$

where $E(h_1, \dots, h_N)$ is the same polynomial function of N variables $h_1, \dots, h_N$ as in the previous section. At this point we only note that if the arguments of this function are non-commuting variables, they should be distributed in the expression in that order in which they are written. The product of non-linear operators will be understood hereafter as their composition.

It is easy to see that the non-linear spectral equation

$$\hat{H} \left(\mathbf{q}, \mathrm{i}\hbar \frac{\partial}{\partial \mathbf{q}} \right) \Psi(\mathbf{q}) = E\Psi(\mathbf{q}), \tag{3.7}$$

is separable by construction and its complete solution (i.e. a solution parametrized by N spectral parameters) has the form

$$E = E(h_1, \dots, h_N), \quad \Psi(\mathbf{q}) = \prod_{\alpha=1}^{N} \Xi_{i[\alpha]}(q_\alpha, h_1, \dots, h_N). \tag{3.8}$$

This immediately follows from the fact that, after fixing the form of the function $E(h_1, \dots, h_N)$, the equation (3.7) becomes equivalent to the separable system (3.4).

So, we have demonstrated that any N arbitrarily chosen functions $F_i(x, y, z_1, \dots, z_N)$, $r = 1, \dots N$ generate a certain, in general, non-linear separable spectral equation by means of the procedure which we shall refer to as the *quantum inverse procedure of separation of variables*. The equation (3.7) will be called the (non-linear) Schrödinger equation.

Now we show that the operators $\hat{H}_\alpha = \hat{H}_\alpha(\mathbf{q}, \mathbf{p})$ can (in some sense) be considered as integrals of motion of a certain completely integrable non-linear "quantum" system. Indeed, from the definition of these operators it immediately follows that they are homogeneous operators of order one, i.e., for any constant c

$$\hat{H}_\alpha(c\Psi) = c\hat{H}_\alpha\Psi. \tag{3.9}$$

Moreover, as follows from (3.5), these operators have a common set of eigenfunctions:

$$\hat{H}_\alpha\Psi = h_\alpha\Psi, \quad \alpha = 1, \dots, N. \tag{3.10}$$

By using these two properties, we can consider two chains of equalities:

$$\hat{H}_\alpha \hat{H}_\beta \Psi = \hat{H}_\alpha h_\beta \Psi = h_\beta \hat{H}_\alpha \Psi = h_\beta h_\alpha \Psi \tag{3.11}$$

and

$$\hat{H}_\beta \hat{H}_\alpha \Psi = \hat{H}_\beta h_\alpha \Psi = h_\alpha \hat{H}_\beta \Psi = h_\alpha h_\beta \Psi. \tag{3.12}$$

Subtracting (3.12) from (3.11), we find that

$$[\hat{H}_\alpha, \hat{H}_\beta]\Psi = 0. \tag{3.13}$$

Note that the operators $\hat{H}_\alpha$ are non-linear only if the multi-parameter spectral equations (3.2) depend on their spectral parameters non-linearly. In the case of linear dependence on spectral parameters the operators $\hat{H}_\alpha$ will obviously be linear. Assume that the set of solutions of equation (3.7) is complete in the sense that W (see above) is a Hilbert space and any function from W can be expanded in solutions of equation (3.7). If the operators $\hat{H}_\alpha$ are linear, then we can claim that equality (3.13) holds for all elements of W and thus the operators $\hat{H}_\alpha$ commute in the strong operator sense. In the case of non-linear operators $\hat{H}_\alpha$ this reasoning does not work, and the commutativity should be understood in the in the weak sense, i.e. on the solutions of the spectral problem (3.7).

4 The non-linear quantization method

In the previous section we used the term "Schrödinger equation" and the adjective "quantum" in order to stress the fact that the equations (2.7) and (3.7) are related to each other by some "quantization procedure". This means that taking in the "quantum Schrödinger equation" (3.7) the classical limit $\hbar \to 0$, we obtain the classical Hamilton–Jacobi equation (2.7). In order to see this, it is sufficient to represent the "wavefunctions" $\Psi(\mathbf{q})$ in the form

$$\Psi(\mathbf{q}) = \exp\left(\frac{\mathrm{i}}{\hbar} S(\mathbf{q})\right), \tag{4.1}$$

after which it becomes clear that the factorizability of the wavefunctions implies the decomposability of their logarithms, i.e. classical actions. This enables one to write

$$\Psi_i(q_i) = \exp\left(\frac{\mathrm{i}}{\hbar} S_i(q_i)\right), \quad i = 1, \ldots, N \tag{4.2}$$

which reduces the problem of verifying our assertion to checking that in the classical limit the multi-parameter spectral equations (3.1) reduce to the equations (2.1). But this is obvious because the substitution of (4.2) into (3.1) gives an expression whose constant term in $\hbar$ exactly coincides with (2.1). All other terms of higher orders in $\hbar$ vanish in the limit $\hbar \to 0$.

5 An example

In this section we consider a simple example demonstrating how does the proposed scheme work. We start with two F-functions of the form

$$F_1(x, y, h, g) = x^2 + y^2 - \frac{\epsilon}{2a^4} h^2 - \left(\frac{2}{a^2} x^2 + 1\right) h + g \tag{5.1}$$

and

$$F_2(x,y,h,g) = x^2 + y^2 - \frac{\epsilon}{2a^4}h^2 - \left(\frac{2}{a^2}x^2 + 1\right)h - g. \tag{5.2}$$

in which the variables x and y are associated with classical or quantum coordinates and momenta and h, g are spectral parameters. We see that, generally, the functions (5.1) and (5.2) depend on the spectral parameters non-linearly, but if $\epsilon = 0$ or $a = \infty$, this dependence becomes linear. For this reason ϵ and a will play the role of non-linear deformation parameters of a system. Let us now construct classical and quantum models associated with functions (5.1) and (5.2).

5.1 Classical case

According to the results of section 2, functions (5.1) and (5.2) can be used for building the integrals of motion for a certain two-dimensional completely integrable classical dynamical system. Denoting these integrals by $H = H(\mathbf{q}, \mathbf{p})$ and $G = G(\mathbf{q}, \mathbf{p})$, where $\mathbf{q} = \{q_1, q_2\}$ and $\mathbf{p} = \{p_1, p_2\}$, and following the general prescriptions of section 2, we can write for them the following two elementary solvable equations

$$p_1^2 + q_1^2 - \frac{\epsilon}{2a^4}H^2 - \left(\frac{2}{a^2}q_1^2 + 1\right)H + G = 0 \tag{5.3}$$

and

$$p_2^2 + q_2^2 - \frac{\epsilon}{2a^4}H^2 - \left(\frac{2}{a^2}q_2^2 + 1\right)H - G = 0. \tag{5.4}$$

Let us take the first integral H as the hamiltonian of a system. Define for this hamiltonian the *potential* $V(\mathbf{q}) = H(\mathbf{q}, 0)$. Despite the fact that the hamiltonian cannot be generally represented as the sum of kinetic and potential energies, this function still reflects some general spectral properties of the model. Indeed, as it follows from equations (5.2) and (5.3), the hamiltonian $H = H(\mathbf{q}, \mathbf{p})$ is an increasing function of $\mathbf{p}^2$. Therefore, we have the inequality $H(\mathbf{q}, \mathbf{p}) \geq V(\mathbf{q})$ which means that the volume of the phase space bounded by the level surface $H(\mathbf{q}, \mathbf{p}) = h$ should be finite if $h \leq \max V(\mathbf{q})$ and infinite if $h \geq \max V(\mathbf{q})$.

In the particular case, when the parameter a^4 is very large, the system (5.3) – (5.4) becomes linear, and, solving it, we obtain the model with hamiltonian

$$H = \frac{\mathbf{p}^2 + \mathbf{q}^2}{2}, \tag{5.5}$$

which is nothing else than the two-dimensional symmetric harmonic oscillator, i.e., the sum of hamiltonians of one-dimensional harmonic oscillators. The second integral of motion, G, reduces in this case to the difference of hamiltonians of one-dimensional harmonic oscillators, and thus, the commutativity of H and G is obvious. The potential $V(\mathbf{q}) = \mathbf{q}^2/2$ of this model does not have an upper bound and the motion in this system is always finite.

If a is finite but ϵ is small, then we still have a linear system of equations for H and G, whose solution leads to a little bit more comlicated model with hamiltonian

$$H = \frac{a^2}{2}\frac{\mathbf{p}^2 + \mathbf{q}^2}{a^2 + \mathbf{q}^2}. \tag{5.6}$$

Although the commutativity of H with the corresponding G is not so obvious as in the previous case, it also can be checked by direct computation of the Poisson bracket. Now the potential

of the model is bounded by $\max V(\mathbf{q}) = a^2/2$, and therefore the model (5.6) describes finite motion if $h < a^2/2$ and infinite motion if $h > a^2/2$.

In the most general case, when both ϵ and a are finite numbers, the equations (5.3) – (5.4) become quadratic and have now two different solutions for H. For definiteness we choose that solution which is continuously connected with particular solutions (5.5) and (5.6). It has the form

$$H = \frac{a^2}{\epsilon^2}(\mathbf{q}^2 + a^2)\left\{-1 + \sqrt{1 + \frac{\epsilon^2(\mathbf{p}^2 + \mathbf{q}^2)}{(a^2 + \mathbf{q}^2)^2}}\right\}. \tag{5.7}$$

Reconstructing the second solution G of this system, and checking its commutativity with H we can again make sure that the obtained model is completely integrable. As before, its potential is monotonically increasing function of $\mathbf{q}^2$ and tends to $\max V(\mathbf{q}) = a^2/2$ if $\mathbf{q}^2 \to \infty$.

Despite the fact that the model (5.7) looks rather complicated, its Hamilton – Jacobi equation is separable by substitution $S(\mathbf{q}) = S_1(q_1) + S_2(q)$, and reduces to the system of two one-dimensional equations for $S_1(q_1)$ and $S_2(q_2)$. This system is nothing else than the initial system (5.3) – (5.4) with $p_1 = \partial S_1(q_1)/\partial q_1$, $p_2 = \partial S_2(q_2)/\partial q_2$, $H = h$ and $G = g$. Solving it, we obtain the complete integral of the model

$$S(\mathbf{q}) = \int dq_1 \sqrt{\left(\frac{2h}{a^2} - 1\right) q_1^2 + \frac{\epsilon}{2a^e}h^2 + h - g} + \int dq_2 \sqrt{\left(\frac{2h}{a^2} - 1\right) q_2^2 + \frac{\epsilon}{2a^4}h^2 + h + g}, \tag{5.8}$$

parametrized by two arbitrary parameters h and g. From this solution it is clearly seen that for $h < a^2/2$ the motion in the system is finite, in full accordance with general reasonings given above.

Concluding the exposition of the classical case, let us stress again the fact that in this case there is no principal difference between the models associated with functions (5.1) and (5.2) with linear and non-linear dependence on spectral parameters. We have seen that in non-linear case the models turned out to be more complicated than in the linear one, but they are still ordinary classical models admitting quite standard interpretation in terms of Hamilton – Jacobi equation.

5.2 Quantum case

Let us now use functions (5.1) and (5.2) for building quantum versions of the models discussed in the previous subsection. We start with multi-parameter spectral equations associated with these functions and having the form:

$$\left\{-\hbar^2\frac{\partial^2}{\partial q_1^2} + q_1^2 - \frac{\epsilon}{2a^4}h^2 - \left(\frac{2}{a^2}q_1^2 + 1\right)h + g\right\}\Psi_1(q_1) = 0 \tag{5.9}$$

and

$$\left\{-\hbar^2\frac{\partial^2}{\partial q_2^2} + q_2^2 - \frac{\epsilon}{2a^4}h^2 - \left(\frac{2}{a^2}q_2^2 + 1\right)h - g\right\}\Psi_2(q_1) = 0 \tag{5.10}$$

It is not difficult to see that the system of these equations will have discrete spectrum if we require the square integrability of both functions $\Psi_1(q_1)$ and $\Psi_2(q_2)$. Indeed, let us rewrite these equations in the form

$$\left\{-\hbar^2\frac{\partial^2}{\partial q_1^2} + \omega^2 q_1^2\right\}\Psi_1(q_1) = e_1\Psi_1(q_1) \tag{5.11}$$

and

$$\left\{-\hbar^2\frac{\partial^2}{\partial q_2^2}+\omega^2 q_2^2\right\}\Psi_2(q_2)=e_2\Psi_2(q_2) \tag{5.12}$$

with

$$\omega=\sqrt{1-\frac{2h}{a^2}} \tag{5.13}$$

and

$$e_1=\frac{\epsilon}{2a^4}h^2+h-g, \qquad e_2=\frac{\epsilon}{2a^4}h^2+h+g. \tag{5.14}$$

We see that formulas (5.11) and (5.12) look as ordinary Schrödinger equations for simple harmonic oscillators, and therefore, if the functions $\Psi_1(q_1)$ and $\Psi_2(q_2)$ are normalizable, have standard solutions

$$\Psi_1(q_1)=\mathcal{H}_n(\sqrt{\omega/\hbar}q_1)\exp\left\{\frac{\omega q_1^2}{\hbar}\right\}, \qquad \Psi_1(q_2)=\mathcal{H}_m(\sqrt{\omega/\hbar}q_2)\exp\left\{\frac{\omega q_2^2}{\hbar}\right\} \tag{5.15}$$

and

$$e_1=\omega\hbar(2n+1), \qquad e_2=\omega\hbar(2m+1), \tag{5.16}$$

where by $\mathcal{H}_n$ and $\mathcal{H}_m$ we denoted the ordinary Hermite polynomials. Comparing formula (5.16) with (5.13) and (5.14), we obtain the system of two equations

$$\frac{\epsilon}{2a^4}h^2+h-g=(2n+1)\hbar\sqrt{1-\frac{2h}{a^2}}, \qquad \frac{\epsilon}{2a^4}h^2+h+g=(2m+1)\hbar\sqrt{1-\frac{2h}{a^2}} \tag{5.17}$$

for h and g. The sum of these equations gives a single equation for h:

$$\frac{\epsilon}{2a^4}h^2+h=(n+m+1)\hbar\sqrt{1-\frac{2h}{a^2}}, \tag{5.18}$$

which, obviously, has a discrete set of solutions.

Consider particular cases of equation (5.16). Let a be large. Then the solution of this equation can be written down immediately. It is

$$h=(n+m+1)\hbar. \tag{5.19}$$

We see that the spectrum of the parameter h is unbounded. Let now a be finite and ϵ be small. Then we obtain a quadratic equation for h. Solving it and choosing that solution which is a continuous deformation of (5.19), we obtain

$$h=\frac{(n+m+1)^2\hbar^2}{a^2}\left\{-1+\sqrt{1+\frac{a^4}{(n+m+1)^2\hbar^2}}\right\}. \tag{5.20}$$

In this case the spectrum of the parameter h is bounded by $h_{max}=a^2/2$. If the numbers n, m are large, then the distances between neighbouring spectral points are small, so that the point

$h_{max} = a^2/2$ is the accumulation point of the spectrum. The general case of arbitrary ϵ and a can be considered analogously. The explicit solution of equation (5.18) is very complicated, because (5.18) is now fourth order algebraic equation. However, the qualitative behaviour of the spectrum is the same as in the last case. As before, the spectrum is bounded by $h_{max} = a^2/2$ and this is its accumulation point.

Thus, we have demonstrated the normalizability of functions $\Psi_1(q_1)$ and $\Psi_2(q_2)$ as well as the discreteness and infiniteness of the spectrum of equations (5.9) and (5.10) in the interval $0 < h < a^2/2$. Analogously, it can be shown that for $h > a^2/2$ the functions $\Psi_1(q_1)$ and $\Psi_2(q_2)$ are non-normalizable and the spectrum of multi-parameter spectral equations (5.9) and (5.10) is continuous, in full accordance with the classical case.

Let us now try to understand whether the spectral values of the parameter h can be considered as the eigenvalues of a certain two-dimensional operator in Hilbert space. In order to do this, it is sufficient to follow the general prescriptions of section 3. Let us multiply the first equation (5.9) by $\Psi_2(q_2)$, and the second one, (5.10), by $\Psi_1(q_1)$. Introducing the function $\Psi(\mathbf{q}) = \Psi_1(q_1)\Psi_2(q_2)$ and adding the equations (5.9) and (5.10), we get a single equation for $\Psi(\mathbf{q})$,

$$\left\{-\hbar^2\Delta + \mathbf{q}^2 - \frac{\epsilon}{a^4}h^2 - 2\left(\frac{1}{a^2}\mathbf{q}^2 + 1\right)h\right\}\Psi(\mathbf{q}) = 0, \tag{5.21}$$

containing a single spectral parameter h. Consider first the particular cases of this equation.

Let a be large. Then the terms proportional to h^2 and $h\mathbf{q}^2$ in (5.21) vanish and we obtain the ordinary Schrödinger equation for a two-dimensional harmonic oscillator

$$\frac{1}{2}\left\{-\hbar^2\Delta + \mathbf{q}^2\right\}\Psi(\mathbf{q}) = h\Psi(\mathbf{q}). \tag{5.22}$$

Let now a be finite and ϵ small. Then the only term proportional to h^2 in (5.21) vanishes and we obtain again a linear spectral equation but with the weight function $\mathbf{q}^2/a^2+1$. In order to reduce the hamiltonian of this equation to the hermitean form, we can redefine the wavefunctions as $\Psi(\mathbf{q}) \to (\mathbf{q}^2/a^2 + 1)^{1/2}\Psi(\mathbf{q})$, after which the equation becomes

$$\frac{a^2}{2}\frac{1}{\sqrt{\mathbf{q}^2 + a^2}}\left\{-\hbar^2\Delta + \mathbf{q}^2\right\}\frac{1}{\sqrt{\mathbf{q}^2 + a^2}}\Psi(\mathbf{q}) = h\Psi(\mathbf{q}). \tag{5.23}$$

Let us now consider the general case of arbitrary a and ϵ. Now the spectral parameter h does not enter in equation (5.21) linearly and thus, cannot be considered as an eigenvalue of some linear operator. The only thing what we can do in this case, is to solve the equation (5.21) with respect to h, and then, multiplying the result by $\Psi(\mathbf{q})$ and exchanging the left- and right-hand sides, write the nonlinear differential equation

$$\frac{a^2}{\epsilon^2}(\mathbf{q}^2 + a^2)\left\{-1 + \sqrt{1 + \frac{\epsilon^2\left(-\hbar^2\Delta\Psi(\mathbf{q}) + \mathbf{q}^2\Psi(\mathbf{q})\right)}{(a^2 + \mathbf{q}^2)^2\Psi(\mathbf{q})}}\right\}\Psi(\mathbf{q}) = h\Psi(\mathbf{q}) \tag{5.24}$$

Note that the ordinary (linear) quantization of the classical system with hamiltonian H leads to the linear Schrödinger equation of the form

$$\frac{a^2}{\epsilon^2}(\mathbf{q}^2 + a^2)\left\{-1 + \sqrt{1 + \frac{\epsilon^2\left(-\hbar^2\Delta + \mathbf{q}^2\right)}{(a^2 + \mathbf{q}^2)^2}}\right\}\Psi(\mathbf{q}) = h\Psi(\mathbf{q}) \tag{5.25}$$

with all necessary permutations of non-commuting operators guaranteeing the hermitian symmetry of the corresponding hamiltonian. Expanding equations (5.24) and (5.25) in $\hbar^2$, it is easy to see that the first two terms of these expansions coincide. The difference (the non-linearity) appears only in the third term and is of order $\hbar^2$. This non-linearity is of the type $(\Delta\Psi/\Psi)^2\Psi$ and is very similar to non-linearities appearing in Doebner – Goldin quantization scheme.

6 Conclusion

Of course, the example we discussed here, is rather artifical and hardly has some relation to reality (even if one belives that quantum world is non-linear), but it clearly demonstrates the idea lying in the ground of our approach. At any rate, irrespective of the physical reasonability of models obtainable by means of our non-linear quantization procedure, the method which we proposed provides a constructive way for extending the notion of quantum integrability to non-linear spectral problems and gives a practical tool for buiding completely integrable non-linear spectral equations in Hilbert space.

An interesting application of this method would be construction of non-linear versions of quantum Gaudin models associated with higher Lie algebras. As it was noted in ref. [13], there exist multi-parameter spectral equations whose solutions exactly coincide with Bethe ansatz solutions for the Gaudin models. Some of these equations depend on spectral parameters linearly. The application of quantum IMSV to such equations leads to ordinary (linear) Gaudin models (the simplest $sl(2)$ case of such a transformation was considered in detail in ref. [12]). However, if the rank of a Lie algebra is suffuciently high, then, along with the "linear" multi-parameter spectral equations, there are equations in which the spectral parameters enter non-linearly. The application of quantum IMSV to such equations should lead to completely integrable models having the same spectra as the ordinary Gaudin models but realized by non-linear operators in Hilbert space. An explicit construction of such models is an interesting mathematical problem and we hope to consider it in one of the fortcoming publications.

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